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ON THE DIFFERENTIATION OF AN INFINITE SERIES
TERM BY TERM.

BY M. B. PORTER.

If

$$f(x) = \sum_1^{\infty} u_n(x), \quad a < x < b,$$

denote a convergent series of single valued, differentiable, real functions of the real variable x , the ordinary sufficient condition that the derivative $f'(x)$ exist and that

$$f'(x) = \sum_1^{\infty} u'_n(x),$$

namely, that each term $u_n(x)$ be differentiable and that the series $\sum_1^{\infty} u'_n(x)$ be uniformly convergent, is practically limited to the case where the uniformity can be tested by means of a so called M -series. That is, we shall have

$$f'(x) = \sum_1^{\infty} u'_n(x), \quad a < x < b,$$

if there exist a convergent series of positive terms, $\sum_1^{\infty} M_n$, such that

$$|u'_n(x)| \leq M_n, \quad a < x < b. \tag{A}$$

The practical importance of this test in that, in particular, it is immediately applicable to power series and thus leads to a simple as well as rigorous treatment of undetermined coefficients, renders an elementary proof of it desirable. The following proof, while probably not new,* has not come to the notice of the writer elsewhere, and seems to be sufficiently elementary.

If x and $x + \Delta x$ lie in the interval of convergence of the series

$$f(x) = \sum_1^{\infty} u_n(x),$$

* Dini, *Functionen einer reellen Grösse*, p. 154, uses the Mean Value Theorem, but as he deduces a more general criterion, his proof is much less elementary.

(19)

then, by the primary definition of convergence, we have

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \sum_{n=1}^{\infty} \frac{u_n(x + \Delta x) - u_n(x)}{\Delta x},$$

and, by the first Mean Value Theorem,

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \sum_{n=1}^{\infty} u'_n(x + \theta_n \Delta x), \quad 0 < \theta_n < 1. \quad (1)$$

If ϵ be an arbitrarily assigned positive number as small as we please, (A) tells us that a positive integer N can be found such that

$$\left| \sum_{N+1}^{\infty} u'_n(x + \theta_n \Delta x) \right| < \frac{\epsilon}{2}. \quad (2)$$

Furthermore, since each term $u_n(x)$ has a derivative $u'_n(x)$, a number δ can be found such that, if $|\Delta x| < \delta$,

$$\frac{u_n(x + \Delta x) - u_n(x)}{\Delta x} = u'_n(x + \theta_n \Delta x)$$

will differ from its limit, $u'_n(x)$, numerically by less than $\epsilon/2n$, so that

$$\left| \sum_{1}^{n} u'_n(x + \theta_n \Delta x) - \sum_{1}^{n} u'_n(x) \right| < \frac{\epsilon}{2}. \quad (3)$$

Hence it follows that

$$\left| \frac{f(x + \Delta x) - f(x)}{\Delta x} - \sum_{1}^{\infty} u'_n(x) \right| < \epsilon, \quad |\Delta x| < \delta,$$

which proves the theorem.

The continuity, or even the integrability, of the derivatives $u'_n(x)$ is not postulated in the above proof, since the Mean Value Theorem merely requires that the derivatives exist.

The same proof can be applied when x denotes the complex variable, $x_1 + x_2 i$, the Mean Value Theorem* for functions of a complex variable now being employed.

NEW HAVEN, CONNECTICUT,
AUGUST, 1901.

* Stolz, *Diff. u. Integralrechnung*, vol. 2, p. 95.